

Solvable groups of interval exchange transformations

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Abstract

We prove that any finitely generated torsion free solvable subgroup of the group IET of all Interval Exchange Transformations is virtually abelian. In contrast, the lamplighter groups $A \wr \mathbb{Z}^k$ embed in IET for every finite abelian group A , and we construct uncountably many non pairwise isomorphic 3-step solvable subgroups of IET as semi-direct products of a lamplighter group with an abelian group.

We also prove that for every non-abelian finite group F , the group $F \wr \mathbb{Z}^k$ does not embed in IET.

The group IET and its subgroups

The group IET of interval exchange transformations is the group of all bijections of the interval $[0, 1)$ that are piecewise translations with finitely many discontinuity points.

Rather unexpectedly, the recent study of this group has given evidences that it is not as big as one could have thought, in several commonly accepted features. For instance, in [2] we established that IET does not have many free subgroups (if any at all), and that the connected Lie groups that can embed in it are only abelian. Another fact in this direction is that given any finitely generated subgroup of IET, and any point $x \in [0, 1)$ the orbit of x grows (in cardinality) at most polynomially in the word length of the elements of the subgroup ([2, 2.6]).

Yet another instance of these evidences is given by the main result of [7] of Juschenko and Monod which implies that certain natural subgroups of this group contain only amenable subgroups. More precisely, given $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the subgroup of IET_α of transformations whose translation lengths are all multiples of α modulo 1 is amenable. Indeed, given any finitely generated subgroup G of IET_α , G can be viewed as a group of homeomorphisms of the cantor set K obtained by blowing up the R_α -orbit of the discontinuity points of the generators of G , where R_α is the rotation $x \mapsto x + \alpha \pmod 1$ ([1]). Denoting by \hat{R}_α the homeomorphism of K induced by R_α , this embeds G in the full topological group of \hat{R}_α , which is amenable by [7]. This has been extended in [6] to subgroups of rational rank ≤ 2 , ie such that the subgroup of \mathbb{Q}/\mathbb{Z} generated by the translation lengths of its elements does not contain \mathbb{Z}^3 .

Given these evidences, we chose to investigate the possible solvable subgroups of IET.

Results

In order to describe elementary examples of subgroups of IET, let us enlarge a bit the context, and instead of interval exchange transformations on the interval $[0, 1)$ we consider the group $\text{IET}(\mathcal{D})$ of interval exchange transformations on a domain \mathcal{D} consisting of a disjoint union of finitely many oriented circles, and oriented half-open intervals, closed on the left (see Section 1.1). This does not make a real difference as $\text{IET} \simeq \text{IET}(\mathcal{D})$.

Now for all $n \in \mathbb{N}$, \mathbb{Z}^n embeds in $\text{IET}(\mathbb{R}/\mathbb{Z})$ as a group of rotations. The following general simple fact then implies that every finitely generated virtually abelian group embeds in IET.

Proposition 1 (Proposition 1.2). *Let G be a group, and assume that some finite index subgroup of G embeds in IET. Then so does G .*

It is then natural to ask which virtually polycyclic groups embed in IET. Our first result shows that only virtually abelian ones do.

Theorem 2 (See Corollary 3.6). *Let H be a virtually polycyclic group. Then H embeds into IET if and only if it is virtually abelian.*

Since a polycyclic group is virtually torsion-free, this result is in fact a corollary of the following theorem which applies to all torsion-free solvable subgroups of IET.

Theorem 3 (Theorem 3.5). *Every finitely generated torsion-free solvable subgroup of IET is virtually abelian.*

Now if we allow torsion, a much greater variety of subgroups exists. The first interesting example is an embedding of the lamplighter group $L = (\mathbb{Z}/n\mathbb{Z}) \wr \mathbb{Z}^k$ in IET. Note that this group L is solvable (in fact metabelian), has exponential growth, and is not virtually torsion-free.

To describe this embedding, consider the domain $\mathcal{D} = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, a disjoint union of n circles. Choose $\Lambda \subset \mathbb{R}/\mathbb{Z}$ a subgroup isomorphic to \mathbb{Z}^k , and view Λ as a group of *synchronized rotations* in $\text{IET}(\mathcal{D})$, i. e. by making $\theta \in \Lambda$ act on $\mathcal{D} = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ by $(i, x) \mapsto (i, x + \theta)$. Consider the interval $I = [0, 1/2) \subset \mathbb{R}/\mathbb{Z}$, and let τ be the transformation that is the identity outside $(\mathbb{Z}/n\mathbb{Z}) \times I$, and sends (i, x) to $(i + 1, x)$ for $x \in I$. Then the subgroup of $\text{IET}(\mathcal{D})$ generated by τ and Λ is isomorphic to L . This is illustrated in Figure 1. More generally, taking $\mathcal{D} = A \times (\mathbb{R}/\mathbb{Z})$ for some finite abelian group A , this construction yields the following result:

Proposition 4 (see Propositions 4.1 and 4.2). *For any finite abelian group A and any $k \geq 1$, the wreath product $L = A \wr \mathbb{Z}^k$ embeds in IET.*

One could try a similar construction, replacing the abelian finite group A by a non-abelian one. But the group obtained would not be the wreath product. In fact, such a wreath product cannot embed in IET as the following result shows.

Theorem 5 (Theorem 4.4). *If F is a finite group and if $F \wr \mathbb{Z}$ embeds as a subgroup in IET, then F is abelian.*

It would be interesting to know which finitely generated wreath products embed in IET.

Starting from a subgroup $G < \text{IET}(\mathcal{D})$ and a finite abelian group A , the construction above allows to construct groups of $\text{IET}(\mathcal{D} \times A)$ isomorphic to $G \ltimes \mathcal{F}_A$ where \mathcal{F}_A is a subgroup of the abelian group $A^{\mathcal{D}}$. We then prove that, in contrast with the torsion-free case, this construction yields a huge variety of isomorphism classes of solvable subgroups in IET.

Theorem 6 (Theorem 4.9). *There exists uncountably many isomorphism classes of subgroups of IET that are generated by 3 elements, and that are solvable of derived length 3.*

The method we use consists of embedding many semidirect products in a way that is related to the twisted embeddings used in [6].

About proofs

The proof of Theorem 4.4 saying that $F \wr \mathbb{Z}$ does not embed into IET uses the fact that orbits in $[0, 1)$ by a finitely generated subgroup of IET have polynomial growth. On the other hand, if $F \wr \mathbb{Z} = (\oplus_{n \in \mathbb{Z}} F) \rtimes \mathbb{Z}$ embeds in IET, F and its conjugate have to commute with each other. This gives strong algebraic restrictions on the action on F . Using Birkhoff theorem, we show that if F itself is non-commutative, then the orbit growth of $F \wr \mathbb{Z}$ has to be exponential, a contradiction.

To prove that there are uncountably many groups as in Theorem 6, we start with a lamplighter group $G = (\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$ constructed above on $\mathcal{D} = (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, where \mathbb{Z} acts on the three circles by setting a generator to act as a synchronized rotation on them with irrational angle α . Then, we consider $\mathcal{D}' = (\mathbb{Z}/2\mathbb{Z}) \times \mathcal{D}$, on which we make G act diagonally. Then we choose an interval J in $\{0\} \times (\mathbb{R}/\mathbb{Z}) \subset \mathcal{D}$ and define τ_J on \mathcal{D}' by the identity outside $(\mathbb{Z}/2\mathbb{Z}) \times J$, and by $(i, x) \mapsto (i + 1, x)$ on $(\mathbb{Z}/3\mathbb{Z}) \times J \subset \mathcal{D}'$. This is illustrated in Figure 2. The group H generated by G and τ_J is a homomorphic image of the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr G$, but this is not an embedding. Still, the group H generated by G and τ_J has the structure of a semidirect product $G \ltimes \mathcal{F}$ where \mathcal{F} is an infinite abelian group of exponent 2. Using Birkhoff ergodic theorem, we prove that from the isomorphism class of H , one can read off the length of J modulo a countable additive group. This proves that by varying length of J , we get uncountably many distinct isomorphism classes of groups H .

The classification of torsion-free solvable subgroups of IET in Theorem 3 is based on the fact that centralizers of a *minimal* interval exchange transformation T is small. Indeed, if T is an irrational rotation on a circle, then its centralizer consists of the whole group of rotations on this circle; and if T is not conjugate to such a rotation, a theorem by Novak shows that its centralizer is virtually cyclic. Now if T is not minimal, its centralizer can be much larger: it will for instance contain a group isomorphic to IET if T fixes a non-empty subinterval.

Thus, we need to understand the orbit closures of a finitely generated group, and we also need to understand how it varies when we pass to a subgroup of finite index.

A result by Imanishi about the holonomy of codimension 1 foliations shows that for each finitely generated group $G < \text{IET}(\mathcal{D})$, there is a partition of \mathcal{D} into finitely many G -invariant subdomains, such that in restriction to each subdomain, either every orbit is dense (such a subdomain is called an *irreducible component*), or every orbit is finite of the same cardinal (and one can say more, see Proposition 2.3). In particular, for each $x \in \mathcal{D}$, $\overline{G.x}$ is either finite, or is the closure of a subdomain (not a Cantor set).

When passing to a finite index subgroup G_0 of G , it could happen that an irreducible component for G splits into several irreducible components for G_0 .

For example, consider $\mathcal{D} = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$, and consider the subgroup G of $\text{IET}(\mathcal{D})$ generated by the three following transformations τ, R_0, R_1 . Let τ be the involution $(i, x) \mapsto (i + 1, x)$; let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and let R_0 be the rotation of angle α on the circle $\{0\} \times (\mathbb{R}/\mathbb{Z})$ and as the identity on the circle $\{1\} \times (\mathbb{R}/\mathbb{Z})$; and let $R_1 = \tau R_0 \tau^{-1}$. Then $\langle R_0, R_1 \rangle \simeq \mathbb{Z}^2$, and $G = \langle \tau, R_0 \rangle \simeq (\mathbb{Z} \oplus \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})$, and any orbit of G is dense.

However, $G_0 = \langle R_0, R_1 \rangle$ is a finite index subgroup which preserves each circle. The two circles are the irreducible components of G_0 . But this phenomenon cannot occur any more when passing to a further finite index G_1 of G_0 because G_1 has to contain an irrational rotation on each circle, thus ensuring that the two circles are still irreducible components of G_1 . This group G_0 is what we call stable. The following technical result of independent interest shows that this is a general fact.

Theorem 7 (see Theorem 2.9). *Given a finitely generated group $G < \text{IET}$, there exists a finite index subgroup G_0 which is stable in the following sense:*

1. G_0 acts by the identity outside its irreducible components

2. If G_1 is a finite index of G_0 , any irreducible component of G_0 is also an irreducible component of G_1 .

1 Generalities

1.1 Definitions

Definition 1.1. *In all the following, a domain will be a non-empty disjoint union of finitely many oriented circles, and oriented half-open, bounded intervals (closed on the left).*

Given a domain \mathcal{D} , the group $\text{IET}(\mathcal{D})$ of interval exchange transformations of \mathcal{D} is the group of bijections of \mathcal{D} that are orientation preserving piecewise isometries, left continuous with finitely many discontinuity points.

By convention, we define $\text{IET} = \text{IET}([0, 1))$.

Given two disjoint domains $\mathcal{D}_1, \mathcal{D}_2$ having the same total length, there is an element of $\text{IET}(\mathcal{D}_1 \sqcup \mathcal{D}_2)$ that sends \mathcal{D}_1 on \mathcal{D}_2 . We call such an element an interval exchange bijection from \mathcal{D}_1 to \mathcal{D}_2 . This element then conjugates $\text{IET}(\mathcal{D}_2)$ to $\text{IET}(\mathcal{D}_1)$. Observe also that rescaling a domain doesn't change its group of interval exchange transformations. In particular, for any domains $\mathcal{D}_1, \mathcal{D}_2$, $\text{IET}(\mathcal{D}_1)$ and $\text{IET}(\mathcal{D}_2)$ are always isomorphic.

1.2 Finite extensions

Proposition 1.2. *Let G be a group, and assume that some finite index subgroup of G embeds in IET . Then so does G .*

Proof. Without loss of generality, consider $H < G$ a normal subgroup of finite index that embeds in $\text{IET}(\mathcal{D})$ for some domain \mathcal{D} . Let Q be the finite quotient $Q = G/H$. It is a classical algebraic fact (see [9]) that G embeds in the wreath product $H \wr Q = H^Q \rtimes Q$ (where Q acts on H^Q by permuting coordinates). Thus, it suffices to show that $H^Q \rtimes Q$ embeds in IET .

Consider the domain $\mathcal{D}' = Q \times \mathcal{D}$, and embed H^Q in $\text{IET}(\mathcal{D}')$ by making $(h_q)_{q \in Q}$ act on $\mathcal{D}' = Q \times \mathcal{D}$ by $(q, x) \mapsto (q, h_q.x)$. Then Q acts on \mathcal{D}' by left multiplication on the left coordinate. This naturally extends to a morphism $H^Q \rtimes Q \rightarrow \text{IET}(\mathcal{D}')$ which is clearly one-to-one. \square

2 Irreducibility and stability for finitely generated subgroups of IET

2.1 IET and irreducibility

Let \mathcal{D} be a domain. Let $G = \langle S \rangle$ be a finitely generated subgroup of the group $\text{IET}(\mathcal{D})$ of interval exchange transformations on \mathcal{D} .

If \mathcal{D}_0 is a subdomain of \mathcal{D} that is invariant by the action of G , then G naturally maps to $\text{IET}(\mathcal{D}_0)$ by restriction. Moreover, if $\mathcal{D} = \mathcal{D}_0 \sqcup \mathcal{D}_1$ where both subdomains are invariant by G , then the induced morphism from G to $\text{IET}(\mathcal{D}_0) \times \text{IET}(\mathcal{D}_1)$ is injective.

Definition 2.1 (Irreducibility.). *We say that G is irreducible (on \mathcal{D}) if every G -orbit in \mathcal{D} is dense in \mathcal{D} , or equivalently, if no subdomain of \mathcal{D} is invariant under G .*

We say that a subdomain J of \mathcal{D} is an irreducible component for G if it is G -invariant, and if G restricted to J is irreducible.

If $s \in \text{IET}(\mathcal{D})$, we denote by $\text{Disc}(s)$ the set of discontinuity points of s . If S is a set of elements of $\text{IET}(\mathcal{D})$, $\text{Disc}(S) \subset \mathring{\mathcal{D}}$ is the set of discontinuity points of elements of S .

We say that $x, y \in \mathring{\mathcal{D}}$ are in the same *regular orbit* for a symmetric set $S \subset \text{IET}(\mathcal{D})$ if there exists a word $w = g_n \dots g_1$ written in the alphabet S (i.e. $\forall i, g_i \in S$) such that $y = w(x)$, and for all $i < n$, $(g_i \dots g_1)(x) \in \mathring{\mathcal{D}} \setminus \text{Disc}(g_{i+1})$. Under such circumstances, there exists $\epsilon > 0$ such that for all $i \leq n$, $(g_i \dots g_1)$ is continuous on $]x - \epsilon, x + \epsilon[$.

By convention, the regular orbit of a point $x \in \partial\mathcal{D}$ is $\{x\}$. We may write $\text{Reg}(x, S^*)$ to denote the set of points that are in the same regular orbit of x for S .

Let us denote by $D_f(S)$ the set of points of discontinuity of the elements of S whose regular orbit is finite.

We say that we cut a domain \mathcal{D}_1 along a finite set of points if we map it to some domain \mathcal{D}_2 by a interval exchange bijection $\tau : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ that is discontinuous exactly on this set of points, and such that τ^{-1} is continuous. For instance, if $\mathcal{D}_1 = [a, b)$, cutting along $c \in [a, b)$ yields a domain $\mathcal{D}_2 = [a, c) \sqcup [c, b)$, and if \mathcal{D}_1 is a circle, cutting along one point yields a domain that is a half-open interval. If $G_1 \subset \text{IET}(\mathcal{D}_1)$, we then get by conjugation a group $G_2 \subset \text{IET}(\mathcal{D}_2)$.

Remark 2.2. Observe that if $G < \text{IET}(\mathcal{D})$ there is always a domain \mathcal{D}' and $G' < \text{IET}(\mathcal{D}')$ conjugate to G by an interval exchange bijection from \mathcal{D} to \mathcal{D}' such that $D_f(S') = \emptyset$ (with S' the conjugate of S). Indeed, it suffices to cut \mathcal{D} along all (finitely many) points that are in the regular orbit of a point in $D_f(S)$. See [3].

Let us recall the version of Imanishi decomposition theorem that we will need.

Proposition 2.3 (Imanishi theorem [5], see Th. 3.1 in [3]). *Let $G < \text{IET}(\mathcal{D})$ be a finitely generated group, and S a finite symmetric generating set. Assume that $D_f(S)$ is empty (in other words, for all $x \in \text{Disc}(S)$, $\text{Reg}(x, S^*)$ is infinite).*

Then for each connected component I of \mathcal{D} , one of the two following assertions is true.

1. *$G.I$ is a union of connected components of \mathcal{D} on which G acts with finite orbits all of the same cardinal; the restriction to $G.I$ of any $g \in G$ is continuous, and in particular, G permutes the connected components of $G.I$.*
2. *$G.I$ is a union of connected components of \mathcal{D} , and the action of G on $G.I$ is irreducible.*

□

Remark 2.4. Theorem 3.1 of [3] is stated in terms of systems of isometries between closed subintervals of a finite union of compact intervals. If $\overline{\mathcal{D}}$ denote the obvious compactification of \mathcal{D} , then each element of S defines finitely many isometries between closed intervals of $\overline{\mathcal{D}}$ to which one can apply this result. Our assumption $D_f(S) = \emptyset$ amounts to asking that the set E appearing in their statement satisfies $E = \partial\overline{\mathcal{D}}$. Since all elements IET preserve the orientation, there cannot be any twisted family of finite orbits in the sense of [3, Th 3.1]. Our assertion 1 includes the possibility that $G.I$ is a disjoint union of circles, and that the finite index subgroup preserving I acts as a finite group of rotations. This possibility does not appear in [3, Th 3.1] because \mathcal{D} is a union of intervals there.

Let us record the following corollary.

Corollary 2.5. *If $G < \text{IET}(\mathcal{D})$ is finitely generated, the domain \mathcal{D} is the disjoint union of finitely many G -invariant subdomains in restriction to which G acts as a finite group, or is irreducible.*

In other words, there exists a unique finite collection of irreducible components of G , $\text{Irred}(G) = \{I_1, \dots, I_r\}$ which are disjoint subdomains of \mathcal{D} , and which satisfy that G acts as a finite group on the G -invariant subdomain $\mathcal{D} \setminus (\bigcup_i I_i)$. □

Proof. One can conjugate the group G to a subgroup G' of $\text{IET}(\mathcal{D}')$ on another domain \mathcal{D}' obtained by Remark 2.2, in which $D_f(S') = \emptyset$ and apply Imanishi theorem on G' .

The connected components of G' are in bijection with some subdomains of \mathcal{D} . On the subdomains corresponding to components upon which G' acts as a finite group, G acts as a finite group, and on those corresponding to components upon which G' is irreducible, G is irreducible, as it was noted in Remark 2.8. \square

Note that whereas $\text{Irred}(G) = \{I_1, \dots, I_r\}$ is uniquely defined, there is no uniqueness in general in the decomposition of the complement $\mathcal{D} \setminus (\bigcup_i I_i)$ into G -invariant subdomains upon which G acts as a finite group.

2.2 A stability for finitely generated groups of IET

Let G be a finite generated subgroup of $\text{IET}(\mathcal{D})$.

Definition 2.6 (Stability). *We say that G is stable if for any subdomain $J \subset \mathcal{D}$ which is invariant by a finite index subgroup of G , J is G -invariant.*

Here is an equivalent definition.

Lemma 2.7. *The subgroup G is stable if each of its finite orbits is trivial, and if, for every irreducible component J of G , and every finite index subgroup H of G , the restriction of H on J is irreducible.*

Proof. Let $I_1, \dots, I_r \subset \mathcal{D}$ be the irreducible components of G , and $I' = \mathcal{D} \setminus (I_1 \cup \dots \cup I_r)$.

Assume that G is stable. If G has a non-trivial finite orbit (necessarily in I'), then there exists a subdomain $J \subset I'$ and $g \in G$ such that $gJ \neq J$. Then J is a subdomain that is invariant under the finite index subgroup of G acting trivially on I' , but not G -invariant, a contradiction. If some finite index subgroup $G_0 < G$ does not act irreducibly on some I_i , then there is a G_0 -invariant subdomain $J \subsetneq I_i$ and J is not G -invariant because I_i is an irreducible component of G , a contradiction.

Conversely, assume that the statement in the lemma holds and let $J \subset \mathcal{D}$ be a subdomain invariant under a finite index subgroup $G_0 < G$. Since G acts trivially on I' , $J \cap I'$ is G -invariant. Now for each i , $J \cap I_i$ is G_0 -invariant, and since by assumption G_0 acts irreducibly on I_i , we either get that $J \cap I_i$ is empty or $J = I_i$, in particular, $J \cap I_i$ is G -invariant. Since this holds for each i , J is G -invariant. \square

Remark 2.8. If \mathcal{D} and \mathcal{D}' are two domains with a interval exchange bijection τ from \mathcal{D} to \mathcal{D}' , then a group $G < \text{IET}(\mathcal{D})$ is stable (resp. irreducible) if and only if its conjugate by τ in $\text{IET}(\mathcal{D}')$ is stable (resp. irreducible).

We will need to prove the following decomposition result.

Theorem 2.9. *If $G < \text{IET}(\mathcal{D})$ is a finitely generated subgroup of $\text{IET}(\mathcal{D})$, it admits a finite index subgroup that is stable.*

Proof of Theorem 2.9. After cutting \mathcal{D} as in the remark 2.2, one can assume that $D_f(S) = \emptyset$. Then, one can apply Imanishi theorem for G : \mathcal{D} decomposes into $\mathcal{D}_{fin} \cup \mathcal{D}_\infty$, for two subdomains \mathcal{D}_{fin} and \mathcal{D}_∞ that are preserved by G , for which $G|_{\mathcal{D}_{fin}} < \text{IET}(\mathcal{D}_{fin})$ is finite, and $\mathcal{D}_\infty = I_1 \cup \dots \cup I_r$ where I_i is G -invariant, and every orbit is dense in I_i .

After taking a finite index subgroup of G , one can assume that $G|_{\mathcal{D}_{fin}}$ is trivial, and therefore the map $G \rightarrow G|_{\mathcal{D}_\infty}$ is an isomorphism. Set $I_0 = \mathcal{D}_{fin}$, and in the following, to keep readable notations, we restrict \mathcal{D} to \mathcal{D}_∞ .

In the rest of the proof we will want to apply Imanishi theorem to all finite index subgroups of G , without subdividing further \mathcal{D} (in particular, this prevents using again Remark 2.2). Since we will not increase the number of connected components of \mathcal{D} , we will be able to find that the irreducible components of a sequence of finite index subgroups of G eventually stabilise.

Lemma 2.10. *Assume $D_f(S) = \emptyset$. Then for all finite index subgroup $G_0 < G$, there exists a symmetric generating set S_0 for which $D_f(S_0) = \emptyset$.*

Proof. We have to find S_0 such that for any $x \in \mathring{\mathcal{D}}$ such that $\text{Reg}(x, S^*)$ is infinite, we have that $\text{Reg}(x, S_0^*)$ is infinite. First, note that this is automatic if the G -orbit of x contains no discontinuity point of S . Indeed, it then follows that for any symmetric generating set S_0 of G_0 , $\text{Reg}(x, S_0^*)$ is $G_0.x$ which is infinite.

Given $x \in \text{Disc}(S)$ we claim that we can find a symmetric generating set S_x of G_0 such that $\text{Reg}(y, S_x^*)$ is infinite for all $y \in G.x \cap \mathring{\mathcal{D}}$. This will conclude the proof since one can take for S_0 the union of all sets S_x for $x \in \text{Disc}(S)$. Let Γ be the orbit graph of x , i. e. the graph whose vertices are the points in $G.x$ and there is an edge labelled $s \in S$ between y and z if $z = sy$. Since S is finite, this is a connected locally finite graph. Let Γ_{reg} be the subgraph of Γ whose vertices are the points in $G.x \cap \mathring{\mathcal{D}}$ and with an edge labelled s between y and $z = sy$ if s is continuous at y . The connected components of Γ_{reg} are precisely the sets $\text{Reg}(y, S^*)$, for $y \in G.x$. Since $\text{Disc}(S)$ and $\mathring{\mathcal{D}}$ are finite, Γ_{reg} has finitely many connected components, all of which are infinite since $D_f(S) = \emptyset$.

We now focus on some connected component Λ of Γ_{reg} . We will prove that for each $y \in \Lambda$, one can find a symmetric generating set S_y of G_0 such that $\text{Reg}(y, S_y^*)$ is infinite and contains $(\Lambda \cap G_0.y)$. Since Λ is the finite union of the sets $(\Lambda \cap G_0.y)$, the lemma will follow.

Choose $U \subset G$ a finite set of representatives of right cosets G_0u of G_0 in G with $1 \in U$. Choose $T \subset \Lambda$ be a maximal tree. For each $z \in \Lambda$, denote by w_z the element of G labelled by the path in T joining y to z . Then $w_z(y) = z$ and w_z is continuous at y . Then consider $u_z \in U$ such that $u_z w_z \in G_0$; in particular, $u_z(z) = u_z w_z(y) \in G_0.y$. Let $\Lambda' \subset \Lambda$ be the set of vertices z such that u_z is continuous at z . Since $\text{Disc}(U)$ is finite, $\Lambda \setminus \Lambda'$ is finite, and since Λ is locally finite and connected, Λ' has finitely many connected components, and so does $T \cap \Lambda'$ (here, we view Λ' and $T \cap \Lambda'$ as induced subgraphs of Λ and T respectively).

Let Δ be the set of points the form $u_z(z)$ for $z \in \Lambda'$. Since $1 \in U$ has no discontinuity point, $\Lambda \cap G_0.y \subset \Delta$. Given a symmetric generating set S_y of G_0 , we view Δ as the vertex set of a graph where we put an edge between y'_1 and y'_2 in Δ if there is a generator $s \in S_y$ sending y'_1 to y'_2 and continuous on y'_1 . We claim that we can find S_y so that Δ is connected. This will conclude the proof since this implies that $\Delta \subset \text{Reg}(y, S_y^*)$, so $\text{Reg}(y, S_y^*)$ is infinite and contains $\Lambda \cap G_0.y$.

First take for S_y a finite symmetric generating set of G_0 containing all the elements of the form $u_{z_2} s u_{z_1}^{-1}$ where s is the label of an edge in T joining z_1 to z_2 (this takes finitely many values). Then for any two points $z_1, z_2 \in \Lambda'$ joined by an edge in T , the corresponding points $z'_1 = u_{z_1}(z_1)$ and $z'_2 = u_{z_2}(z_2)$ are connected by an edge in Δ . Since Λ' has finitely many connected components, so does Δ .

Now choose points y_0, y_1, \dots, y_n in each connected component of $T \cap \Lambda'$ with $y_0 = y$, and let $y'_i = u_{y_i}(y_i) \in \Delta$. For all $i \in \{1, \dots, n\}$, let w_i be the label of the path in T joining y to y_i . Then the element $u_{y_i} w_i \in G_0$ sends y to y'_i and is continuous at y . Thus adding the elements $u_{y_1} w_1, \dots, u_{y_n} w_n$ to S_y makes Δ connected which concludes the proof. \square

Let $G_0 < G$ be a finite index subgroup. Let $J_1, \dots, J_{n(G_0)}$ be the irreducible components of G_0 in \mathcal{D} .

By the lemma, there is a finite symmetric generating set S_0 of G_0 such that $D_f(S_0) = \emptyset$. Imanishi theorem then says that each $i \leq n(G_0)$, the irreducible component J_i is a union of connected component of \mathcal{D} . Let $n(\mathcal{D})$ be the number of connected component of \mathcal{D} . The quantity $n(\mathcal{D}) - n(G_0)$ is therefore non-negative, and when we go to a finite index subgroup, this quantity decreases.

Thus taking a finite index subgroup minimizing the quantity $n(\mathcal{D}) - n(G_0)$ gives a stable group.

□

3 Commutation and solvable subgroups

Let \mathcal{D} be a domain.

Lemma 3.1. *Let $a \in \text{IET}(\mathcal{D})$ be stable with irreducible components $\text{Irred}(a) = \{I_1, \dots, I_r\}$. If $g \in \text{IET}(\mathcal{D})$ is such that gag^{-1} commutes with a , then for all i , either $g(I_i)$ is disjoint from I_1, \dots, I_r , or $g(I_i)$ is equal to some I_j .*

Proof. Observe that $\text{Irred}(gag^{-1}) = \{g(I_1), \dots, g(I_r)\}$. For readability we will write $a^g = gag^{-1}$. Since a commutes with a^g , it permutes the collection $\{g(I_1), \dots, g(I_r)\}$, and therefore some power $a^{r!}$ preserves each $g(I_j)$. By stability, any subdomain preserved by $a^{r!}$ is preserved by a , and therefore a itself preserves each $g(I_j)$.

Similarly, a^g preserves each I_j .

For all j , $g(I_j)$ is therefore a union of a subdomain of F and of some irreducible components of $\langle a \rangle$, hence among the I_i . And symmetrically, each I_j is a union of a subdomain of F and of some domains of the form $g(I_i)$.

$g(I_j) \cap I_i$ is a^g -invariant, and also a invariant. If $g(I_j) \cap I_i \neq \emptyset$, then $g(I_j) \cap I_i = g(I_j)$ by irreducibility of a^g on $g(I_j)$. Similarly, $g(I_j) \cap I_i = I_i$ by irreducibility of a on I_i . It follows that if $g(I_j) \cap I_i \neq \emptyset$ then $g(I_j) = I_i$. The claim follows. □

Corollary 3.2. *Let $a \in G < \text{IET}(\mathcal{D})$ be a stable element with irreducible components I_1, \dots, I_n . Assume that for all $g \in G$, gag^{-1} commutes with a .*

Then for each $i \leq n$, $\{g(I_i), g \in G\}$ is a finite collection of disjoint subdomains.

Proof. Let $g, h \in G$. Assume that $g(I_i) \cap h(I_i) \neq \emptyset$. Then, $h^{-1}g(I_i) \cap I_i \neq \emptyset$, and by Lemma 3.1, this implies that $h^{-1}g(I_i) = I_i$, hence $g(I_i) = h(I_i)$. Thus $\{g(I_i), g \in G\}$ is a collection of disjoint subdomains. It is finite because the measure of \mathcal{D} is finite. □

Proposition 3.3. *Let G be a subgroup of $\text{IET}(\mathcal{D})$. If there exists a normal abelian subgroup of G , containing some irreducible and stable element, then G is either abelian or virtually cyclic. In particular, G is virtually abelian.*

Before starting the proof, let us recall the following, from [2]. Let $g \in \text{IET}(\mathcal{D})$ and $d(g)$ be the number of discontinuity points of g on \mathcal{D} . Let $\|g\| = \lim_{n \rightarrow \infty} \frac{1}{n} d(g^n)$. In [2, Coro. 2.5], we proved that $\|g\| = 0$ if and only if g is conjugate to a continuous transformation of some domain \mathcal{D}' consisting only of circles, and one of its powers is a rotation on each circle.

Proof. Let a be such an element in $A \triangleleft G$ (with A abelian). If $\|a\| = 0$, then as we mentioned in the preceding discussion, a is conjugate to a continuous transformation on some \mathcal{D}' , and because it is irreducible and stable, \mathcal{D}' has to be a circle and a is an irrational rotation. Then, by [2, Lemma 1.1], its centralizer is conjugate to the rotation group on \mathcal{D}' , and since G normalises A , [2, Lemma 1.1] says that G must also be conjugate to a group of rotations on \mathcal{D}' , hence abelian.

Assume now that $\|a\| > 0$. Since a is irreducible, [10, Prop. 1.5] implies that its centraliser is virtually cyclic, hence so is A . Therefore A has a finite automorphism group. The group G acts by conjugation on A , and since the automorphism group of A is finite, the kernel of this action has finite index in G . This kernel is contained in the centralizer of a though, so G is virtually cyclic. □

Proposition 3.4. *If $G < \text{IET}(\mathcal{D})$ is irreducible, and contains a normal abelian subgroup $A \triangleleft G$, with an element $a \in A$ of infinite order, then G is virtually abelian.*

In particular, if G is finitely generated, then so is A .

Proof. Up to replacing a by a power, Theorem 2.9 allows us to assume that a is stable. Let I_1, \dots, I_k be its irreducible components, and F the complement of their union, in \mathcal{D} .

By Corollary 3.2, $\{g(I_1), g \in G\}$ is a finite collection of disjoint subdomains. By irreducibility of G , it is a partition of \mathcal{D} , and we thus write $\mathcal{D} = M_1 \sqcup \dots \sqcup M_l$ with $M_1 = I_1$, and $M_i = g_i I_1$. There is a finite index subgroup G_0 of G that preserves each M_i . Let k be such that $a^k \in G_0$. Then, a^k is irreducible and stable on I_1 so Proposition 3.3 ensures that the image $G_0|_{I_1}$ of G_0 in $\text{IET}(I_1)$ (well defined, by taking the restriction) is virtually abelian. Similarly, $(a^k)^{g_i}$ is irreducible on M_i , so G_0 has virtually abelian image in restriction to each M_i .

However, since $\mathcal{D} = M_1 \sqcup \dots \sqcup M_l$, the group G_0 injects in $\prod_j \text{IET}(M_j)$, and we saw that the projection $p_j(G_0)$ of its image in each coordinates is virtually abelian.

Consider the intersection of the preimages of a finite index abelian group of $p_j(G_0)$ for each coordinate. This is an abelian subgroup of finite index in G_0 . \square

Theorem 3.5. *Let $G < \text{IET}(\mathcal{D})$ be a finitely generated torsion free solvable group. Then G is virtually abelian.*

Since virtually polycyclic groups are virtually torsion-free, we get

Corollary 3.6. *Any virtually polycyclic subgroup of IET is virtually abelian.* \square

We actually prove the following lemma first.

Lemma 3.7. *Let $G < \text{IET}(\mathcal{D})$ be a finitely generated group, and assume that it contains a torsion-free abelian normal subgroup A . Then A is finitely generated, and there is a finite index subgroup H in G , such that $A \cap [H, H] = \{1\}$.*

Proof. Let $\text{Irred}(G) = \{I_1, \dots, I_k, I_{k+1}, \dots, I_r\}$ where one has ordered the components so that the image of A in $\text{IET}(I_i)$ is a torsion group for all $i > k$ and contains an infinite order element for $i \leq k$. Let $F = \mathcal{D} \setminus (\bigcup_{i=1}^r I_i)$.

For every $i \leq k$, by Proposition 3.4 we get that the image of G in $\text{IET}(I_i)$ is a virtually abelian group.

In other components, the image of A is a torsion group. Let us denote by G_1 the image of G in $\text{IET}(\bigcup_{i=1}^k I_i)$ and by G_2 the image of G in $\text{IET}(\bigcup_{j=k+1}^r I_j) \cup F$.

This gives an embedding $\iota : G \hookrightarrow G_1 \times G_2$, where G_1 is virtually abelian, and $p_2 \circ \iota(A)$ is torsion. In particular, because A is torsion-free, $p_1 \circ \iota$ is injective in restriction on A .

It already follows that A is finitely generated, since it embeds as a subgroup of a finitely generated virtually abelian group.

Consider H_1 an abelian finite index subgroup in G_1 , and H the preimage of H_1 in G , which is a finite index subgroup.

We saw that the map $p_1 \circ \iota$ is injective on A , but it vanishes on $[H, H]$ because $p_1 \circ \iota(H) = H_1$ is abelian. Therefore $A \cap [H, H] = \{1\}$, thus establishing the lemma. \square

Proof of Theorem 3.5. Consider $G < \text{IET}(\mathcal{D})$ a finitely generated torsion free solvable group. Then its derived series has a largest index n for which $G^{(n)} \neq \{1\}$ (where $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ and $G^{(0)} = G$). If $n = 1$, G is abelian, we thus proceed by induction on n .

Since G is torsion free, $G^{(n)}$ is torsion free, infinite, and it is also abelian and normal in G . We may then apply Lemma 3.7 to $A = G^{(n)}$, to find that there is a finite index subgroup H of G such that $[H, H] \cap G^{(n)} = \{1\}$. However, $H^{(n)} \subset G^{(n)}$, and it follows

that $H^{(n)}$ is trivial. The induction hypothesis implies that H is virtually abelian, hence so is G . □

4 Lamps and lighters

4.1 A lamplighter group in IET

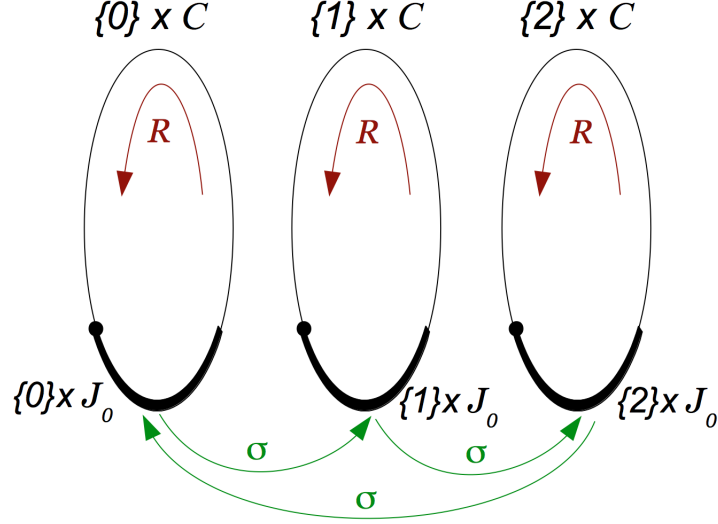


Figure 1: A Lamplighter group $(\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$ in IET. The three circles $\{i\} \times \mathcal{C}$ for $i \in \mathbb{Z}/3\mathbb{Z}$ are visible. The transformation R rotates each circle by the irrational angle θ . The support of the transformation $\sigma = \sigma_{1, J_0}$ is the union of the bold arcs.

Proposition 4.1. *For all finite abelian group A , the group $A \wr \mathbb{Z}$ embeds in IET.*

Recall that in general, the group $A \wr G$ is the group $(\oplus_{i \in G} A) \rtimes G$, where G acts by shifting coordinates: if $g \in G$, and $(a_i)_{i \in G} \in (\oplus_{i \in G} A)$ is an almost null sequence, $g \cdot (a_i)_{i \in G} = (a_{gi})_{i \in G}$.

We will actually describe an embedding in $\text{IET}(\mathcal{D})$ for a certain domain \mathcal{D} . However, IET and $\text{IET}(\mathcal{D})$ are isomorphic, and the choice of \mathcal{D} is only for convenience.

The construction is illustrated in Figure 1.

Proof. Consider the domain $\mathcal{D} = A \times \mathcal{C}$ where $\mathcal{C} = \mathbb{R}/\mathbb{Z}$. Given $a \in A$ and $J \subset \mathcal{C}$ a subinterval, let $\sigma_{a, J}$ be the element of $\text{IET}(\mathcal{D})$ defined for all $(a', x) \in A \times \mathcal{C}$ by

$$\sigma_{a, J} \cdot (a', x) = \begin{cases} (aa', x) & \text{if } x \in J \\ (a', x) & \text{if } x \notin J. \end{cases}$$

Note that the support of $\sigma_{a, J}$ is $A \times J$ when $a \neq 1$. We define \mathcal{A}_J as the subgroup of $\text{IET}(\mathcal{D})$ consisting of the elements $\sigma_{a, J}$ for $a \in A$ (note that \mathcal{A}_J is isomorphic to A as long as J is non-empty).

Now fix $J_0 = [0, 1/2[\subset \mathcal{C}$, and let $\mathcal{A} = \mathcal{A}_{J_0}$. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and let $R \in \text{IET}(\mathcal{D})$ be the rotation by θ on each circle: $R(a, x) = (a, x + \theta)$. We claim that the group generated by R and \mathcal{A} is isomorphic to $A \wr \mathbb{Z}$.

First, one easily checks that for all $J, J' \subset \mathcal{C}$, any element of \mathcal{A}_J commutes with any element of $\mathcal{A}_{J'}$ (this is because A is abelian). Denote by t a generator of the factor \mathbb{Z} in $A \wr \mathbb{Z}$. Since $R^k \mathcal{A} R^{-k} = \mathcal{A}_{R^k J_0}$, there is a homomorphism $\varphi : A \wr \mathbb{Z} \rightarrow \langle R, \mathcal{A} \rangle$ sending t to R and sending the almost null sequence $(a_i)_{i \in \mathbb{Z}}$ to $\prod_{i \in \mathbb{Z}} \sigma_{a_i, R^i J_0}$.

To prove that φ is injective, consider an element $g = ((a_i)_{i \in \mathbb{Z}}, t^k)$ of its kernel. Since $\varphi(g)$ sends (a, x) to some $(a', x + k\theta)$, we get that $k = 0$. This means that $\varphi(g)$ is a commuting product $\varphi(g) = \sigma_{a_1, J_1} \dots \sigma_{a_n, J_n}$ where J_1, \dots, J_n are distinct translates of J_0 by a multiple of θ . We can assume that $n > 0$ and that no a_i is trivial. For all $x \in \mathcal{C}$, consider $\mathcal{J}_x \subset \{1, \dots, n\}$ the set of indices $j \in \{1, \dots, n\}$ such that $x \in J_j$. Note that when x varies, \mathcal{J}_x changes only when x crosses an endpoint of some J_i , and that \mathcal{J}_x then changes by exactly one element (this is because the $2n$ endpoints of the J_i 's are distinct since θ is irrational). Thus, there exist $x, x' \in \mathcal{C}$ and $i_0 \in \{1, \dots, n\}$ such that $\mathcal{J}_{x'} = \mathcal{J}_x \cup \{J_{i_0}\}$. Now, for all $a \in A$, $\varphi(g)(a, x) = (a \prod_{j \in \mathcal{J}_x} a_j, x)$, and since $\varphi(g)$ is the identity, $\prod_{j \in \mathcal{J}_x} a_j = 1$. Similarly, $\prod_{j \in \mathcal{J}_{x'}} a_j = 1$, so $a_{i_0} = 1$, a contradiction. \square

The argument above immediately generalizes to $A \wr \mathbb{Z}^d$, by replacing the rotation of angle θ by d rotations of rationally independant angles. We thus get:

Proposition 4.2. *For all $d \geq 1$, and all finite abelian group A , $A \wr \mathbb{Z}^d$ embeds in IET.*

Remark 4.3. Given $G < \text{IET}$, and A a finite abelian group, we don't know when the group $A \wr G$ embeds in IET.

4.2 Lamps must commute

We now put restrictions on which wreath products may embed in IET. Note that if a group A contains an infinite order element, then $A \wr \mathbb{Z}$ contains the torsion-free solvable group $\mathbb{Z} \wr \mathbb{Z}$ which does not embed in IET by Theorem 3.5.

Theorem 4.4. *Let $L = F \wr \mathbb{Z}$ with F finite non-abelian. Then L does not embed in IET.*

We will use several times the observation (see [2, Lemma 6.2]) that for any finitely generated subgroup G of $\text{IET}(\mathcal{D})$, the orbit of any point of the domain \mathcal{D} has polynomial growth in the following sense: given a finite generating set and the corresponding word metric on G , denoting by B_R the ball of radius R in G , there exists a polynomial P such that for all $x \in \mathcal{D}$ and all $R \geq 0$, $\#(B_R.x) \leq P(R)$.

The theorem will be proved by showing that if L did embed in IET, there would exist an orbit with exponential growth.

If $E \subset [0, 1)$ has positive measure and if T is any element of IET, then by Poincaré recurrence Theorem, the orbit of almost every $x \in E$ comes back to E . We will need a more precise estimate, that comes from Birkhoff theorem.

Lemma 4.5. *Let $E \subset [0, 1)$ have positive measure, and let T be any element in IET.*

Then there exists $x_0 \in [0, 1)$, and constants $\alpha > 0$, $\beta > 0$, such that for all n ,

$$\#\{i \mid 0 \leq i < n, T^i(x) \in E\} \geq \alpha n - \beta.$$

Proof. Assume first that the Lebesgue measure μ is ergodic, and apply Birkhoff theorem to the characteristic function f of E . Then for almost every x , $\frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \rightarrow \mu(E)$, so for n large enough, $\frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) \geq \frac{1}{2}\mu(E)$, so $\#\{i \mid 0 \leq i \leq n, T^i(x) \in E\} \geq \frac{1}{2}\mu(E)n$ for n large enough, and the result holds.

If the Lebesgue measure is not ergodic, Birkhoff theorem still says that the limit exists almost everywhere, and the limit is a T -invariant function $l(x)$ having the same average as f , that is $\int l(x)dx = \mu(E)$ (see [8] for instance). It follows that there exist points where $l(x) \geq \mu(E)/2$, and the proof works the same. \square

Now we prove an abstract algebraic lemma that relates the stabilizer of a point in a product of groups to coordinate-wise stabilizers.

Lemma 4.6. *Consider some groups F_i , and an action of $F = F_1 \times \cdots \times F_n$ on a set X . Let $x \in X$, $\text{Stab}(x)$ its stabilizer in F , and $S_i = F_i \cap \text{Stab}(x)$. Let N_i the normalizer of S_i in F_i .*

Then $\text{Stab}(x) \subset N_1 \times \cdots \times N_n$.

In particular, if no S_i is normal in F_i , then $\#F.x \geq 2^n$.

Proof. Note that $N_1 \times \cdots \times N_n = \cap_i N_F(S_i)$, so we have to prove that for each i , any $g \in \text{Stab}(x)$ normalizes S_i . Now if $g \in \text{Stab}(x)$, then g normalizes both $\text{Stab}(x)$ and F_i (because $F_i \triangleleft F$), hence normalizes their intersection, namely S_i .

Let us prove the last comment. The given assumption says that for all i , $[F_i : N_i] \geq 2$ so $\#F.x = [F : \text{Stab}(x)] \geq 2^n$. \square

To prove the theorem, consider $L = F_0 \wr \mathbb{Z}$ for some finite non-abelian group F_0 . We assume that L embeds in IET and argue towards a contradiction. Let t be a generator of \mathbb{Z} viewed as a subgroup of L , and write $F_i = F^{t^i}$. We equip the group G with the word metric corresponding to the generating set $\{t\} \cup F_0$. The subgroup $F_0 \times F_1 \times \cdots \times F_{n-1}$ is contained in the ball of radius $3n$. We identify L with the corresponding subgroup of IET. Since orbits grow polynomially [2, Lemma 6.2], for any $x \in [0, 1)$, the orbit of x under $F_0 \times \cdots \times F_{n-1}$ has to be bounded by a polynomial in n .

Proposition 4.7. *For all $x \in [0, 1)$, $\text{Stab}_{F_0}(x) \triangleleft F_0$.*

Proof. Otherwise, let $E \subset [0, 1)$ be the set of points where $\text{Stab}_{F_0}(x) \not\triangleleft F_0$. Since F_0 is a finite group, this is a subdomain of $[0, 1)$, and it has positive measure. We apply Birkhoff theorem (in the form of Lemma 4.5), and get that there exists $x \in [0, 1)$, $\alpha, \beta > 0$, such that, for all n , there exists $k_n \geq \alpha n - \beta$, and some indices $0 \leq i_1 < \cdots < i_{k_n} \leq n$ such that $t^{i_j}(x) \in E$. Applying the algebraic Lemma 4.6 to $F = F_{i_1} \times \cdots \times F_{i_{k_n}}$ we get that $\#(F.x) \geq 2^{k_n} \geq 2^{\alpha n - \beta}$. Since F is contained in a ball of linear radius in L , this contradicts polynomial growth of orbits. \square

We now prove another algebraic lemma.

Lemma 4.8. *Consider an action of $F = F_1 \times \cdots \times F_n$ on a set X . Let $x \in X$, $\text{Stab}(x)$ its stabilizer in x , and $S_i = \text{Stab}(x) \cap F_i$ its stabilizer in F_i .*

Assume that $\text{Stab}(x) \triangleleft F$ (in particular $S_i \triangleleft F_i$). Consider $Z(F_i/S_i)$ the center of the quotient group F_i/S_i , and $Q_i = (F_i/S_i)/Z(F_i/S_i)$.

Then the natural epimorphism $F_1 \times \cdots \times F_n \rightarrow Q_1 \times \cdots \times Q_n$ factors through an epimorphism $F/\text{Stab}(x) \twoheadrightarrow Q_1 \times \cdots \times Q_n$.

In particular, if all groups F_i/S_i are non-abelian, each Q_i is non-trivial, and $\#F.x = \#F/\text{Stab}(x) \geq 2^n$.

Proof. Consider $g = (g_1, \dots, g_n) \in \text{Stab}(x)$, and denote by \bar{g}_i the image of $g_i \in F_i/S_i$. We have to prove that \bar{g}_i is central in F_i/S_i , in other words that for all $a \in F_i$, $[g_i, a] \in S_i$. Since $F_i \triangleleft F$, $[g_i, a] = (g_i a g_i^{-1}) a^{-1} \in F_i$, and since $S_i = \text{Stab}(x) \cap F_i$ is normal in F , $[g_i, a] = g_i(a g_i^{-1} a) \in S_i$. The lemma follows. \square

Proof of Theorem 4.4. Consider the subgroup $L' \subset L$ generated by t^k , and $F_0 \times \cdots \times F_{k-1}$. Clearly, $L' \simeq F'_0 \wr \mathbb{Z}$ where $F'_0 = F_0 \times \cdots \times F_{k-1}$. Applying Proposition 4.7 to L' , we get that for all x , and all k , the stabilizer of x in $F_0 \times \cdots \times F_{k-1}$ is normal in $F_0 \times \cdots \times F_{k-1}$.

Let $E \subset [0, 1)$ be the set of points x where $F_0/\text{Stab}(x)$ is non-abelian. If E is empty, any commutator of F_0 acts trivially in $[0, 1)$, so F_0 does not act faithfully, a contradiction. So E is a non-empty subdomain and has positive measure. By Lemma 4.5, there exists

$x \in [0, 1)$, $\alpha, \beta > 0$, such that, for all n , there exists $k_n \geq \alpha n - \beta$, and some indices $0 \leq i_1 < \dots < i_{k_n} \leq n$ such that $t^{i_j}(x) \in E$.

Applying the second algebraic Lemma 4.8 to $F = F_{i_1} \times \dots \times F_{i_{k_n}}$ we get that $\#F.x \geq 2^{\alpha n - \beta}$. This contradicts polynomial growth of orbits as above. \square

4.3 Uncountably many solvable groups, via lamplighter-lighters

We finish by an abundance result among solvable subgroups of IET (necessarily with a lot of torsion in view of Theorem 3.5).

Theorem 4.9. *There exist uncountably many isomorphism classes of 3-generated solvable subgroups of derived length 3 in IET.*

These will be obtained as iterated lamplighter-like constructions. Again we will be free to choose the domain \mathcal{D} for convenience.

For the time being, take \mathcal{D} an arbitrary domain. Let $G < \text{IET}(\mathcal{D})$ and $(A, +)$ a finite abelian group. The following construction generalizes the construction showing that lamplighter groups embed. In this case, G would be the cyclic group generated by an irrational rotation on a circle \mathcal{D} .

Let $\mathcal{F} = A^{\mathcal{D}}$ be the additive group of all functions on \mathcal{D} with values in A . Given $a \in A$ and $J \subset \mathcal{D}$, we denote by $a\mathbb{1}_J \in \mathcal{F}$ the function defined by $a\mathbb{1}_J(x) = a$ if $x \in J$, and $a\mathbb{1}_J(x) = 0$ otherwise. The group G acts on \mathcal{F} by precomposition. Given a subdomain $J \subset \mathcal{D}$, let $\mathcal{F}_J \subset \mathcal{F}$ be the smallest G -invariant additive subgroup containing the functions $a\mathbb{1}_J$ for $a \in A$.

Proposition 4.10. *(Lamplighter-like construction)*

Let G be a subgroup of $\text{IET}(\mathcal{D})$, A a finite abelian group, and $J \subset \mathcal{D}$. Then $\mathcal{F}_J \rtimes G$ embeds in IET.

Remark 4.11. In the particular case of Proposition 4.1, we proved that \mathcal{F}_J is isomorphic to $\oplus_{g \in G} A$. In general, there is a natural morphism from $\oplus_{g \in G} A$ to \mathcal{F}_J , but it might be non-injective. One easily constructs examples using a group G that does not act freely on \mathcal{D} .

Proof. Take $\mathcal{D}' = A \times \mathcal{D}$. Embed G in $\text{IET}(\mathcal{D}')$ by setting for all $g \in G$, $g(a, x) = (a, g(x))$.

For all $b \in A$, consider $\sigma_b \in \text{IET}(\mathcal{D}')$ defined by $\sigma_b(a, x) = (b + a, x)$ if $x \in J$, and by $\sigma_b(a, x) = (a, x)$ if $x \notin J$.

We claim that the subgroup G' generated by G and $\{\sigma_b | b \in A\}$ is isomorphic to $\mathcal{F}_J \rtimes G$.

Consider the subset $H \subset \text{IET}(\mathcal{D}')$ of all $T \in \text{IET}(\mathcal{D}')$ such that there exists some transformation $g \in G$ and some $f \in \mathcal{F}_J$ such that $T(a, x) = (a + f(x), g(x))$. It is clear that H is a subgroup of $\text{IET}(\mathcal{D}')$, that H contains G' , and that $G' = H$. Finally, the fact that $H \simeq \mathcal{F}_J \rtimes G$ is also clear. \square

Let $\mathcal{D}_0 = \mathbb{R}/\mathbb{Z}$, and $I = [0, 1/2[\subset \mathcal{D}_0$. Fix once and for all $\alpha \notin \mathbb{Q}$, ρ_α the rotation of \mathcal{D}_0 of angle α . Let G be the lamplighter group $(\mathbb{Z}/3\mathbb{Z}) \wr \mathbb{Z}$ realised as a subgroup of $\text{IET}(\mathcal{D})$ for $\mathcal{D} = (\mathbb{Z}/3\mathbb{Z}) \times \mathcal{D}_0$ as in Proposition 4.1. We denote by R the rotation $(a, x) \mapsto (a, \rho_\alpha(x))$ on \mathcal{D} .

We now perform this lamplighter-like construction a second time, starting with G acting on \mathcal{D} , with $A = \mathbb{Z}/2\mathbb{Z}$ and with some $J \subset \{0\} \times \mathcal{D}_0 \subset \mathcal{D}$. We recall that \mathcal{F}_J is the subgroup of the group of functions from \mathcal{D} to $A = \mathbb{Z}/2\mathbb{Z}$ as above. Define $H_J = \mathcal{F}_J \rtimes G$ the lamplighter-like group thus obtained, on $\mathcal{D}' = (\mathbb{Z}/2\mathbb{Z}) \times \mathcal{D}$. See Figure 2.

Proposition 4.12. *Assume that $J_1, J_2 \subset \{0\} \times \mathcal{D}_0 \subset \mathcal{D}$, and that $|J_1|, |J_2| < \frac{1}{2}$.*

If $H_{J_1} \simeq H_{J_2}$ then $|J_1| \in \text{Vect}_{\mathbb{Q}}(1, \alpha, a_2, b_2)$, where $a_2, b_2 \in \mathbb{R}/\mathbb{Z}$ are the endpoints of J_2 .

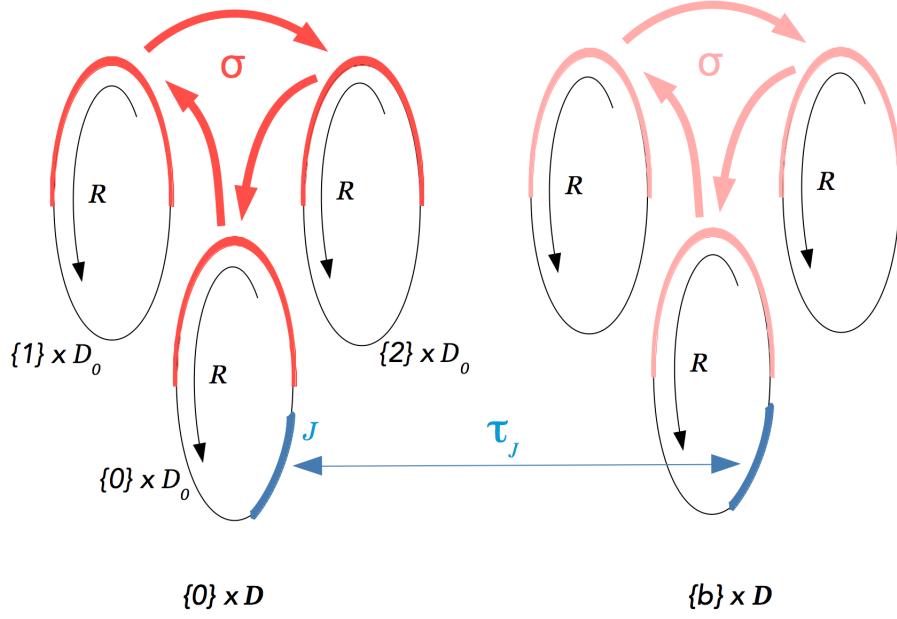


Figure 2: A Lamplighter-like construction in IET producing the examples of Theorem 4.9: on the left, the domain $\mathcal{D} = \mathbb{Z}/3\mathbb{Z} \times \mathcal{D}_0$, and the Lamplighter group $G = \mathbb{Z}/3\mathbb{Z} \ltimes \mathbb{Z}$ generated by a cyclic swap on the three intervals $\mathbb{Z}/3\mathbb{Z} \times I$ (in bold red) and by the simultaneous rotation R . On the right, $\{b\} \times \mathcal{D}$ is the duplicated copy of \mathcal{D} , with the swap τ_J on J and its copy. The group G acts diagonally on $\mathcal{D}' = \mathbb{Z}/2\mathbb{Z} \times \mathcal{D}$. The group $(\oplus_G \mathbb{Z}/2\mathbb{Z}) \rtimes G$ acts on \mathcal{D}' but its image H_J in $\text{IET}(\mathcal{D}')$ depends on J .

Denote by \mathcal{E}_I and $\mathcal{E}_{J_i} \subset \mathbb{R}/\mathbb{Z}$ the orbit under the rotation ρ_α of the endpoints of I and J_i respectively (ie $\mathcal{E}_I = \{0, \frac{1}{2}\} + \alpha\mathbb{Z}$, $\mathcal{E}_{J_i} = \{a_i, b_i\} + \alpha\mathbb{Z}$).

Lemma 4.13. *If $H_{J_1} \simeq H_{J_2}$, then there exist two subdomains $K, K' \subset \mathbb{R}/\mathbb{Z}$ with endpoints in \mathcal{E}_{J_2} and \mathcal{E}_I respectively and $\varepsilon \in \{\pm 1\}$ such that*

$$\forall n \in \mathbb{Z}, \quad \rho_\alpha^n(I) \cap J_1 \neq \emptyset \iff \rho_\alpha^{\varepsilon n}(K) \cap K' \neq \emptyset.$$

Proof of Proposition 4.12 from Lemma 4.13. We have that $\rho_\alpha^n(I) \cap J_1 \neq \emptyset$ if and only if $n\alpha \in J_1 - I$, and that $\rho_\alpha^{\varepsilon n}(K) \cap K' \neq \emptyset$ if and only if $n\alpha \in \varepsilon(K' - K)$.

By Birkhoff theorem, we get that $|J_1 - I| = |K' - K|$. Since $|J_1| < \frac{1}{2} = |I|$, $|J_1 - I| = \frac{1}{2} + |J_1|$.

On the other hand, since $K' - K$ is a union of intervals whose endpoints are in $\mathcal{E}_I - \mathcal{E}_{J_2} \subset \text{Vect}_{\mathbb{Q}}(1, \alpha, a_2, b_2)$, the proposition follows. \square

We view H_J as the semidirect product $\mathcal{F}_J \rtimes G$. Thus each element $h \in H_J$ can thus be written (uniquely) under the form $h = g\tau$ with $\tau \in \mathcal{F}_J$, $g \in G$. Now we view G as a group of interval exchange transformations on $\mathcal{D} = (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$. It is generated by the rotation $R : (a, x) \mapsto (a, x + \alpha)$, and by the lamp element $\sigma \in \mathcal{S}$ (of order 3) that sends $(a, x) \in (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ to $(a + 1, x)$ for $x \in I$, and is the identity otherwise (see left part of figure 2). Since G is a lamplighter group, any $g \in G$ can be written uniquely as $R^n S_f$ where S_f is an IET of the form $(a, x) \mapsto (f(x) + a, x)$ for some function $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$. We denote by $\mathcal{S} = \{S_f\}$ the 3-torsion abelian group of lamps of G . It is freely generated by the $\langle R \rangle$ -conjugates of σ . Thus any $h \in H_J$ is written in a unique way as $h = R^n S_f \tau$ as above.

The kernel of the natural map $H_J \rightarrow \mathbb{Z}$ is the torsion group $N = \mathcal{F}_J \mathcal{S}$. It is exactly the set of elements of finite order.

Denote by b be the generator of $\mathbb{Z}/2\mathbb{Z}$. Given $K \subset \mathcal{D}$ denote by $\tau_K = b\mathbb{1}_K \in \mathcal{F}$. For certain K (for instance $K = J$), $\tau_K \in \mathcal{F}_J$ and thus is in H_J .

Fact 4.14. *If $K \subset \mathcal{D}_0 \times \{0\}$, then $[\sigma, \tau_K] = 1$ if and only if $K \cap I = \emptyset$.*

Proof. From the definition of the semidirect product, $\sigma\tau_K\sigma^{-1} = \tau_K \circ \sigma^{-1} = b\mathbb{1}_{\sigma(K)}$. Hence $[\sigma, \tau_K] = 1$ if and only if $\sigma(K) = K$. If $K \cap I \neq \emptyset$, then $\sigma(K)$ contains a point outside $\mathcal{D}_0 \times \{0\}$, so $\sigma(K) \neq K$. \square

Fact 4.15. *For any $\tau, \tau' \in \mathcal{F}_J$ and any $h \in H_J$, $[h\tau, \tau'] = [h, \tau']$.*

Proof. This is an immediate consequence of the commutation of τ with τ' . \square

Proof of Lemma 4.13. Let $\varphi : H_{J_1} \rightarrow H_{J_2}$ be an isomorphism. We first note that $\mathcal{F}_J \triangleleft H_J$ is precisely the set of elements $g \in H_J$ such that $g^2 = 1$. Moreover, the set of elements of finite order in H_J is the subgroup $N = \mathcal{F}_J\mathcal{S}$ (but there are *exotic* elements of order 3).

Thus $\varphi(\tau_{J_1}) \in \mathcal{F}_{J_2}$ can be viewed as a function $\tau : \mathcal{D} \rightarrow \mathbb{Z}/2\mathbb{Z}$, and $\varphi(\sigma)$ as an element $S'\tau'$ for some $S' \in \mathcal{S}$, and $\tau' \in \mathcal{F}_J$. Now $\varphi(R)$ generates G_{J_2} modulo the torsion subgroup, so \mathcal{F}_{J_2} , so $\varphi(R) = R^\varepsilon S''\tau''$ for some function $\tau'' \in \mathcal{F}_{J_2}$, $S'' \in \mathcal{S}$, and $\varepsilon = \pm 1$.

In what follows, we use the notation gh for hgh^{-1} . Fix $n \in \mathbb{Z}$ and consider the commutator $C = [\sigma, \tau_{J_1}^{R^n}] = [\sigma, \tau_{R^n(J_1)}]$. By Fact 4.14, C is trivial if and only if $R^n(J_1) \cap I = \emptyset$. On the other hand, by Fact 4.15,

$$\varphi(C) = [S'\tau', \tau^{(R^\varepsilon S''\tau'')^n}] = [S', \tau^{(R^\varepsilon S''\tau'')^n}].$$

Now there exists $\tau''' \in \mathcal{F}_{J_2}$ such that $(R^\varepsilon S''\tau'')^n = (R^\varepsilon S'')^n \tau'''$, and since \mathcal{F}_{J_2} is abelian, $\tau^{(R^\varepsilon S''\tau'')^n} = \tau^{(R^\varepsilon S'')^n} \tau'''$. Hence, $\varphi(C) = 1$ if and only if $[S'^{(R^\varepsilon S'')^{-n}}, \tau] = 1$. Now a similar calculation in L shows that $S'^{(R^\varepsilon S'')^{-n}} = S'^{R^{-\varepsilon n} S'''} = S'^{R^{-\varepsilon n}}$, so $\varphi(C) = 1$ if and only if $[S', \tau^{R^{\varepsilon n}}] = 1$.

Given $x \in \mathbb{R}/\mathbb{Z}$, we call the *fiber* of x the 3 point set $F_x = \{(a, x) | a \in \mathbb{Z}/3\mathbb{Z}\} \subset \mathcal{D}$. Let $K \subset \mathbb{R}/\mathbb{Z}$ be the set of points x such that τ is non constant on the fiber of x . Since $\tau \in \mathcal{F}_{J_2}$, K is a union of intervals with endpoints in \mathcal{E}_{J_2} . Write the transformation S' as $(a, x) \mapsto (a + f(x), x)$ for some function $f : \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{Z}/3\mathbb{Z}$, and let K' be the support of f . It is a union of intervals with endpoints in \mathcal{E}_I .

We claim that $[S', \tau^{R^{\varepsilon n}}] = 1$ if and only if $R^{\varepsilon n}(K) \cap K' = \emptyset$. Note that S' preserves each fiber. So fix $x \in \mathbb{R}/\mathbb{Z}$, and F_x its fiber. If $x \notin K'$, then S' acts as the identity on F_x , so $[S', \tau^{R^{\varepsilon n}}]$ fixes F_x . If $x \notin R^{\varepsilon n}(K)$, then $\tau^{R^{\varepsilon n}}$ is a constant function on F_x , and the same conclusion holds. If on the contrary $x \in R^{\varepsilon n}(K) \cap K'$, then $\tau^{R^{\varepsilon n}}$ is not constant on F_x and S acts transitively on F_x so $\tau^{R^{\varepsilon n}} \circ S'$ does not coincide with $\tau^{R^{\varepsilon n}}$ on F_x . This proves the claim and concludes the proof. \square

We can finally prove Theorem 4.9.

Proof of Theorem 4.9. Let $J_x = [0, x[$ for $x < \frac{1}{2}$. Proposition 4.12 shows that given x , there are at most countably many $y < \frac{1}{2}$ such that $H_{J_x} \simeq H_{J_y}$. This gives uncountably many isomorphism classes of groups.

Since G is metabelian, H_{J_x} is solvable of derived length at most 3. Since there are only countably many isomorphism classes of finitely generated metabelian groups, H_{J_x} has derived length exactly 3 for uncountably many values of x . \square

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